Sinusoidal Alternating Waveforms

The terminology *ac* voltage or *ac* current refers to alternating voltage or current. The term alternating indicates only that the waveform alternates between two prescribed levels in a set time sequence. To be absolutely correct, the term sinusoidal, square wave, or triangular must also be applied

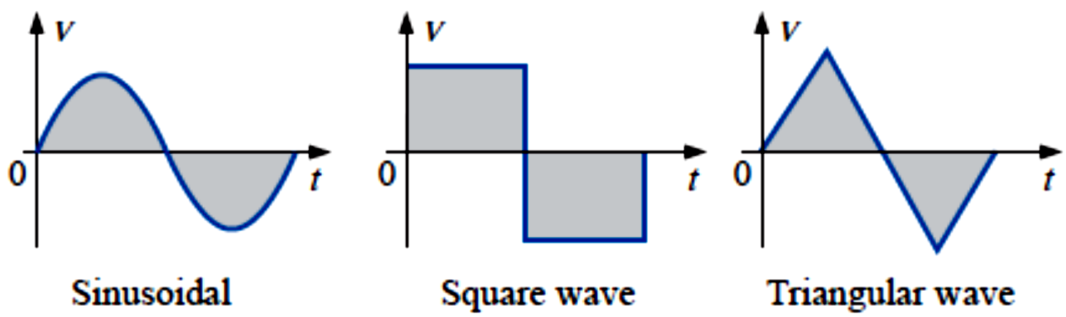


Figure 1: Alternating waveforms.

A sinusoid is a signal that has the form of the sine or cosine function.

A sinusoidal current is usually referred to as alternating current (ac). Such a current reverses at regular time intervals and has alternately positive and negative values. Circuits driven by sinusoidal current or voltage sources are called ac circuits. A sinusoidal signal is easy to generate and transmit. It is the form of voltage generated throughout the world and supplied to homes, factories, laboratories, and so on. It is the dominant form of signal in the communications and electric power industries. Through Fourier analysis, any practical periodic signal can be represented by a sum of sinusoids. Sinusoids, therefore, play an important role in the analysis of periodic signals. Lastly, a sinusoid is easy to handle mathematically. The derivative and integral of a sinusoid are themselves sinusoids. For these and other reasons, the sinusoid is an extremely important function in circuit analysis.

Definitions

- **Periodic waveform:** A waveform that continually repeats itself after the same time interval. The waveform of Fig. 2 is a periodic waveform.
- **Period (T):** The time interval between successive repetitions of a periodic waveform (the period T1 = T2 = T3 in Fig. 2), as long as successive similar points of the periodic waveform are used in determining T.

- **Peak value:** The maximum instantaneous value of a function as measured from the zero-volt level.

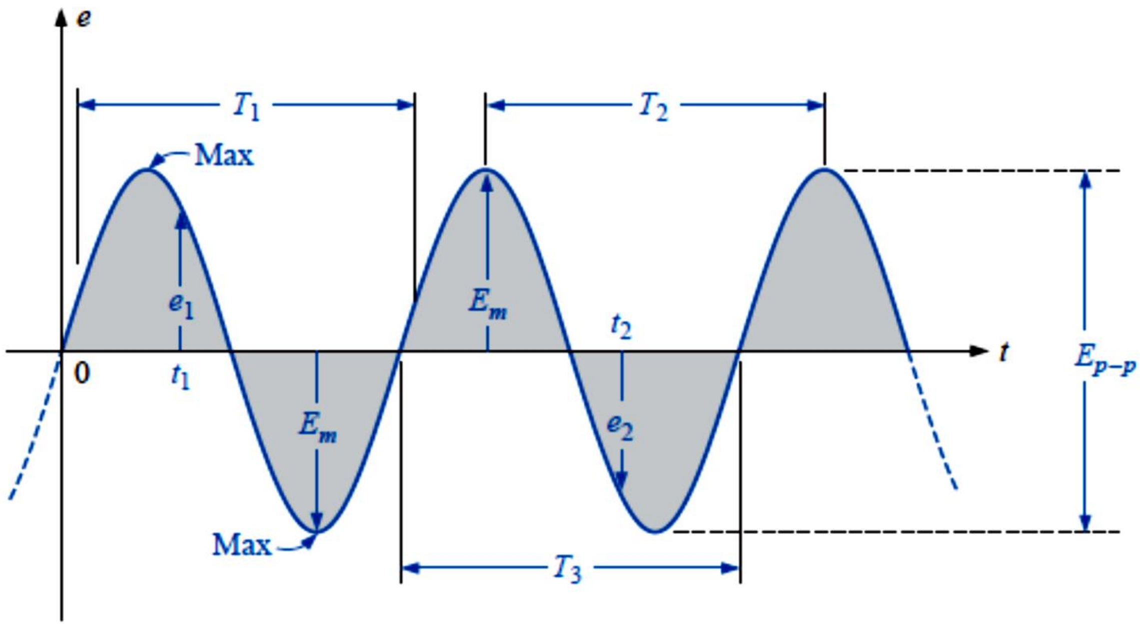


Figure 2: Important parameters for a sinusoidal voltage.

- **Peak-to-peak value:** Denoted by $E_{p\text{-}p}$ or $V_{p\text{-}p}$, the full voltage between positive and negative peaks of the waveform, that is, the sum of the magnitude of the positive and negative peaks.
- Instantaneous value: The magnitude of a waveform at any instant of time; denoted by lowercase letters (e_1, e_2).
- Cycle: The portion of a waveform contained in one period of time.

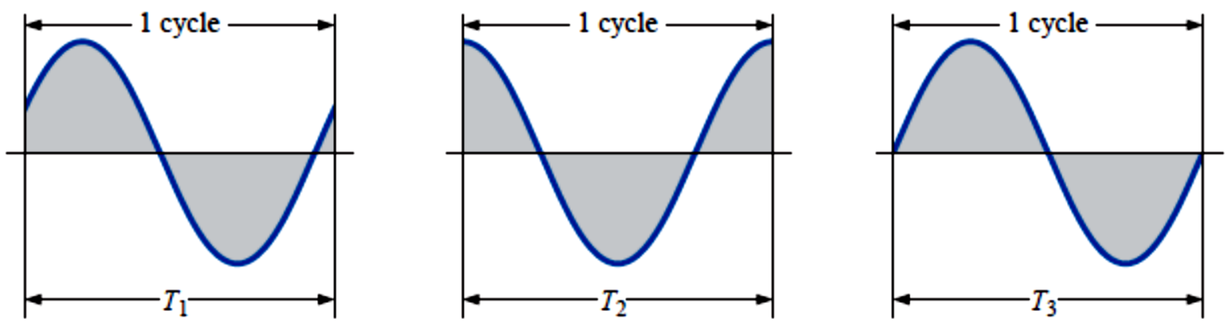


Figure 3: Defining the cycle and period of a sinusoidal waveform.

- **Frequency (f):** The number of cycles that occur in 1 s. The frequency of the waveform of Fig. 4(a) is 1 cycle per second, and for Fig. 4(b), 2 1⁄2 cycles per second. If a waveform of similar shape had a period of 0.5 s [Fig. 4(c)], the frequency would be 2 cycles per second.

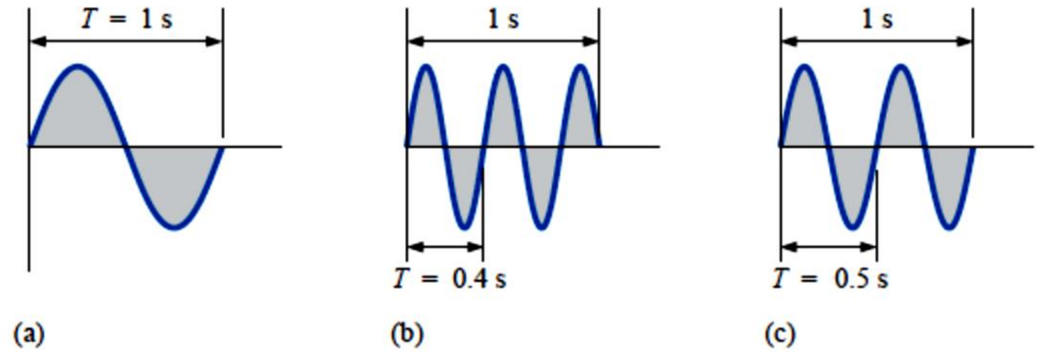


Figure 3: Demonstrating the effect of a changing frequency on the period of a sinusoidal waveform.

The unit of measure for frequency is the *hertz* (Hz), where:

$$\boxed{1 \text{ hertz (Hz)} = 1 \text{ cycle per second (c/s)}}$$

Since the frequency is inversely related to the period—that is, as one increases, the other decreases by an equal amount—the two can be related by the following equation:

$$\boxed{f = \frac{1}{T}} \qquad \begin{aligned} f &= \text{Hz} \\ T &= \text{seconds (s)} \end{aligned}$$

<u>Example 1:</u> Find the period of a periodic waveform with a frequency of:
a. 60 Hz., b. 1000 Hz.

Solutions:

a. $T = \dfrac{1}{f} = \dfrac{1}{60\text{ Hz}} \cong 0.01667 \text{ s or } \mathbf{16.67\ ms}$

b. $T = \dfrac{1}{f} = \dfrac{1}{1000\text{ Hz}} = 10^{-3} \text{ s} = \mathbf{1\ ms}$

<u>Example 2:</u> Determine the frequency of the waveform of Fig. 4.

Solution:

From the figure, T = (25ms - 5ms) = 20ms, and

$$f = \frac{1}{T} = \frac{1}{20 \times 10^{-3} \text{ s}} = \mathbf{50\ Hz}$$

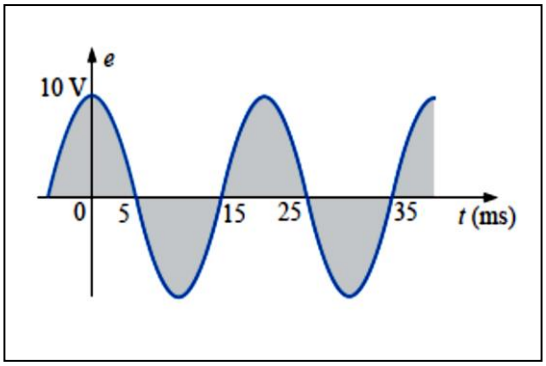


Figure 4: Waveform of Example 2

Example 3: For the pattern of Fig. 5 (**oscilloscope image)** and the indicated sensitivities, determine the period, frequency, and peak value of the waveform.

Solution: One cycle spans 4 divisions. The period is therefore

$$T = 4\,\cancel{\text{div.}}\left(\frac{50\ \mu\text{s}}{\cancel{\text{div.}}}\right) = \mathbf{200\ \mu s}$$

and the frequency is:

$$f = \frac{1}{T} = \frac{1}{200 \times 10^{-6}\ \text{s}} = \mathbf{5\ kHz}$$

The vertical height above the horizontal axis encompasses 2 divisions. Therefore

$$V_m = 2\,\cancel{\text{div.}}\left(\frac{0.1\ \text{V}}{\cancel{\text{div.}}}\right) = \mathbf{0.2\ V}$$

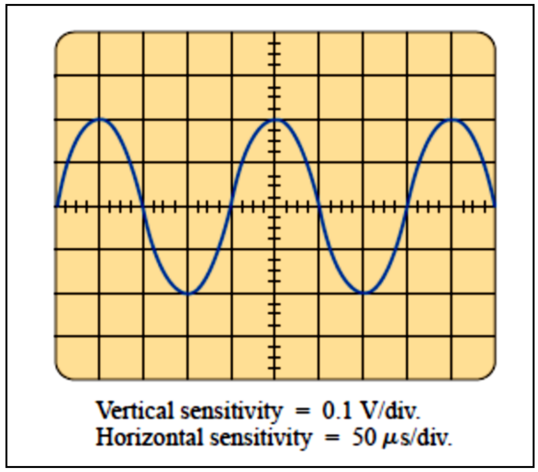


Figure 5: Waveform of Example 3

The sinusoidal waveform is the only alternating waveform whose shape is unaffected by the response characteristics of R, L, and C elements.

One complete set of positive and negative values of an alternating quantity is called a Cycle. Complete cycle is said to spread over 360° or 2π radians.

$\therefore\ 360^\circ = 2\pi$ radians

$$\therefore\ 1\,rad = \frac{360^\circ}{2\pi} = \frac{180^\circ}{\pi}$$

$$\therefore\ 1\,rad = 57.3^\circ$$

To convert from degrees to radians

$$\boxed{\text{Radians} = \left(\frac{\pi}{180^\circ}\right) \times (\text{degrees})}$$

To convert from radian to degree

$$\boxed{\text{Degrees} = \left(\frac{180^\circ}{\pi}\right) \times (\text{radians})}$$

Applying these equations, we find:

$$\mathbf{90^\circ:}\quad \text{Radians} = \frac{\pi}{180^\circ}(90^\circ) = \frac{\pi}{2}\ \mathbf{rad}$$

$$\mathbf{30^\circ:}\quad \text{Radians} = \frac{\pi}{180^\circ}(30^\circ) = \frac{\pi}{6}\ \mathbf{rad}$$

$$\frac{\pi}{3}\ \textbf{rad:}\quad \text{Degrees} = \frac{180^\circ}{\pi}\left(\frac{\pi}{3}\right) = 60^\circ$$

$$\frac{3\pi}{2}\ \textbf{rad:}\quad \text{Degrees} = \frac{180^\circ}{\pi}\left(\frac{3\pi}{2}\right) = 270^\circ$$

In general for the sinusoidal voltage or current:

$e = E_m \sin\theta$, Where θ in degrees

$e = E_m \sin\omega t$, Where $\theta = \omega t$.

$e = E_m \sin 2\pi f t$, Where $\omega = 2\pi f$.

$e = E_m \sin\frac{2\pi}{T}t$, Where $f = \frac{1}{T}$

Phase Relations

If the waveform is shifted to the right or left of 0°, the expression becomes:

$$\boxed{A_m \sin(\omega t \pm \theta)}$$

where θ is the angle in degrees or radians that the waveform has been shifted.
If the waveform passes through the horizontal axis with a positive going (increasing with time) slope before 0°, as shown in Fig. 6, the expression is:

$$\boxed{A_m \sin(\omega t + \theta)}$$

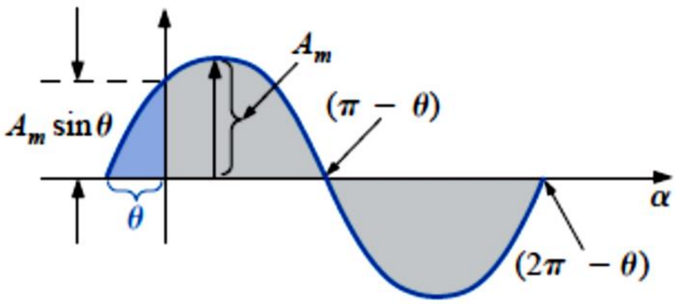


Figure 6:Defining the phase shift for a sinusoidal function that crosses the horizontal axis with a positive slope before 0°. $A_m \sin(\omega t + \theta)$

If the waveform passes through the horizontal axis with a positive-going slope after 0°, as shown in Fig. 7, the expression is:

$$\boxed{A_m \sin(\omega t - \theta)}$$

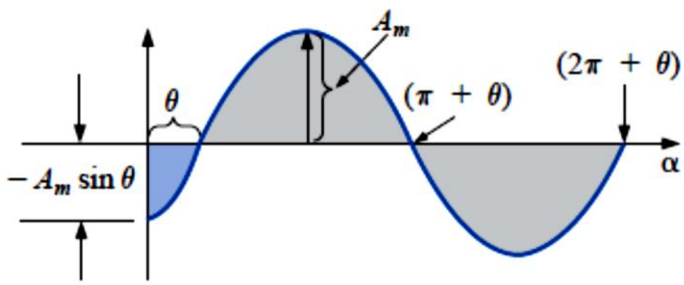


Figure 7:Defining the phase shift for a sinusoidal function that crosses the horizontal axis with a positive slope after 0°. $A_m \sin(\omega t - \theta)$

If the waveform crosses the horizontal axis with a positive-going slope 90° (π/2) sooner, as shown in Fig. 8, it is called a *cosine wave*; that is,

$$\sin(\omega t + 90°) = \sin\left(\omega t + \frac{\pi}{2}\right) = \cos \omega t$$

Or

$$\sin \omega t = \cos(\omega t - 90°) = \cos\left(\omega t - \frac{\pi}{2}\right)$$

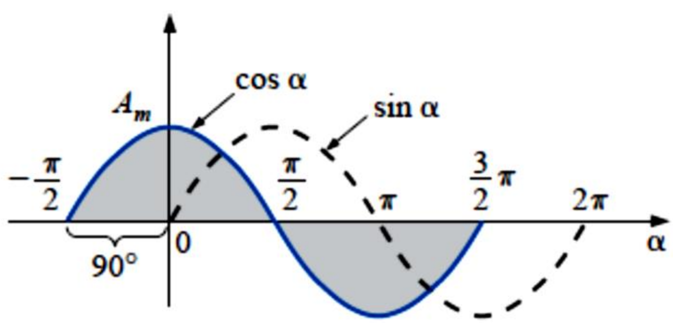


Figure 8: Phase relationship between a sine wave and a cosine wave.

The terms *lead* and *lag* are used to indicate the relationship between two sinusoidal waveforms of the same frequency plotted on the same set of axes.

In Fig. 8, the cosine curve is said to *lead* the sine curve by 90°, and the sine curve is said to *lag* the cosine curve by 90°. The 90° is referred to as the phase angle between the two waveforms.

Some Useful Relations:

$$\cos\alpha = \sin(\alpha + 90°)$$
$$\sin\alpha = \cos(\alpha - 90°)$$
$$-\sin\alpha = \sin(\alpha \pm 180°)$$
$$-\cos\alpha = \sin(\alpha + 270°) = \sin(\alpha - 90°)$$
etc.

$$\sin(-\alpha) = -\sin\alpha$$
$$\cos(-\alpha) = \cos\alpha$$

*The **phase relationship** between two waveforms indicates which one **leads** or **lags**, and by how many degrees or radians.*

Example 4: What is the phase relationship between the sinusoidal waveforms of each of the following sets?

a. $v = 10\sin(\omega t + 30°)$
   $i = 5\sin(\omega t + 70°)$

b. $i = 15\sin(\omega t + 60°)$
   $v = 10\sin(\omega t - 20°)$

c. $i = 2\cos(\omega t + 10°)$
   $v = 3\sin(\omega t - 10°)$

d. $i = -\sin(\omega t + 30°)$
   $v = 2\sin(\omega t + 10°)$

e. $i = -2\cos(\omega t - 60°)$
   $v = 3\sin(\omega t - 150°)$

Solution:

a. See Fig. 9.
***i* leads *v* by 40°, or *v* lags *i* by 40°.**

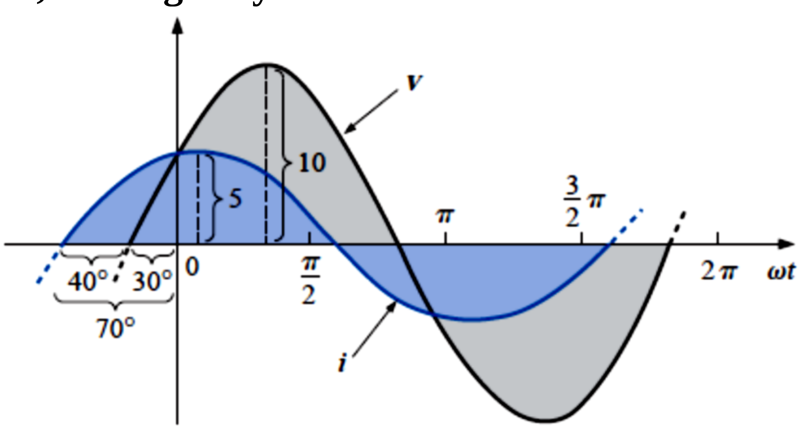


Figure 9: Example 4 (a); i leads v by 40°

b. See Fig. 10.
***i* leads *v* by 80°, or *v* lags *i* by 80°.**

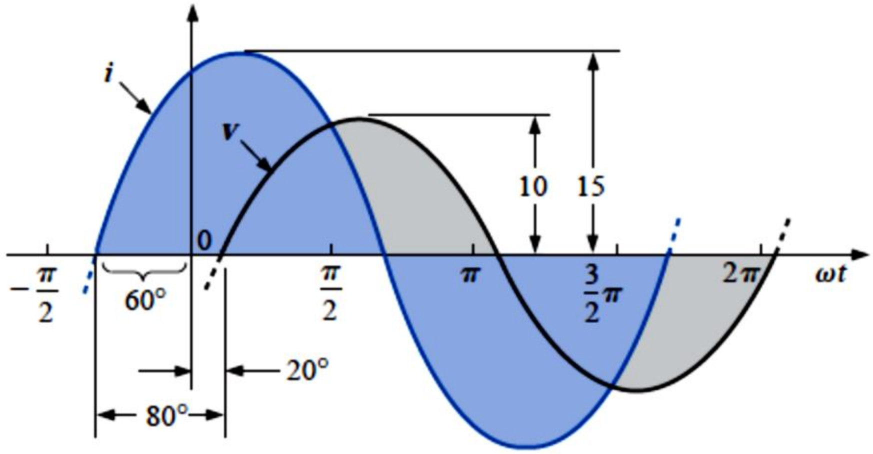


Figure 10: Example 4 (b); *i* leads *v* by 80°

c. See Fig. 11.

$$i = 2\cos(\omega t + 10°) = 2\sin(\omega t + 10° + 90°)$$
$$= 2\sin(\omega t + 100°)$$

***i* leads *v* by 110°, or *v* lags *i* by 110°.**

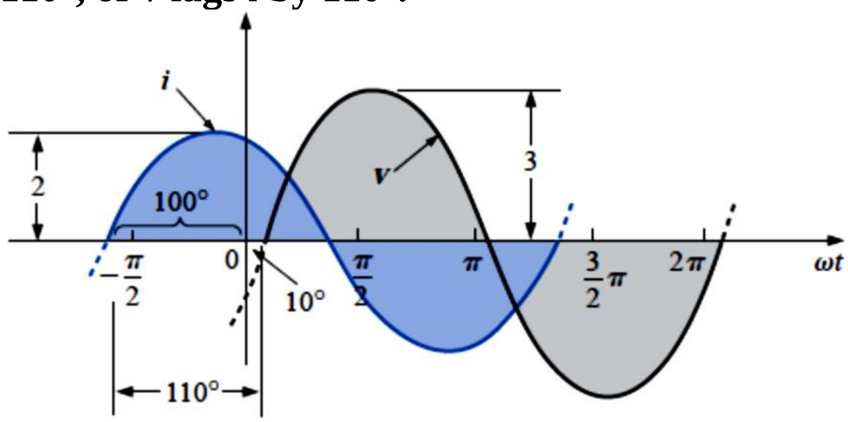


Figure 11: Example 4 (c); *i* leads *v* by 110°

d. See Fig. 12.

$$-\sin(\omega t + 30°) = \sin(\omega t + 30° \overset{\text{Note}}{-} 180°)$$
$$= \sin(\omega t - 150°)$$

***v* leads *i* by 160°, or *i* lags *v* by 160°.**

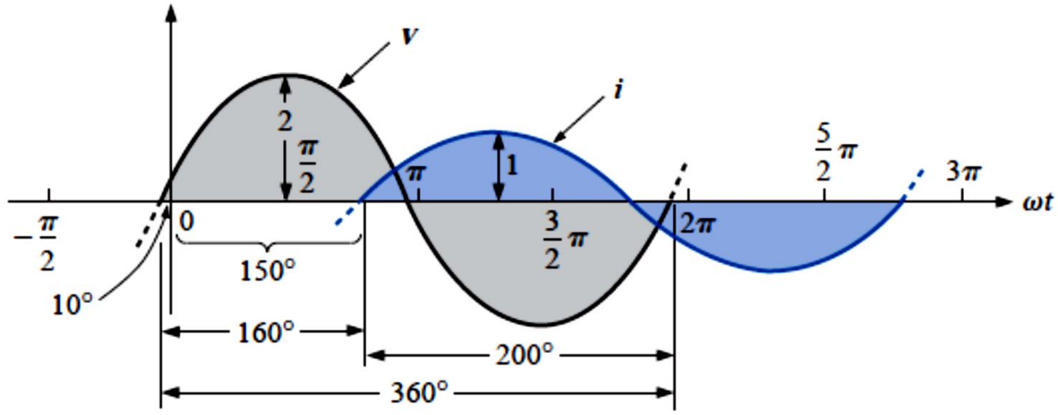


Figure 12: Example 4 (d); *v* leads *i* by 160°

Or using:

$$-\sin(\omega t + 30°) = \sin(\omega t + 30° + \overset{\text{Note}}{180°})$$
$$= \sin(\omega t + 210°)$$

***i* leads *v* by 200°, or *v* lags *i* by 200°.**

e. See Fig. 13.

$$i = -2\cos(\omega t - 60°) = 2\cos(\omega t - 60° - \overset{\text{By choice}}{180°})$$
$$= 2\cos(\omega t - 240°)$$

However, $\cos\alpha = \sin(\alpha + 90°)$

so that $2\cos(\omega t - 240°) = 2\sin(\omega t - 240° + 90°)$

$= 2\sin(\omega t - 150°)$

$\therefore\ i = 2\sin(\omega t - 150°)$

$v = 3\sin(\omega t - 150°)$

***v* and *i* are in phase.**

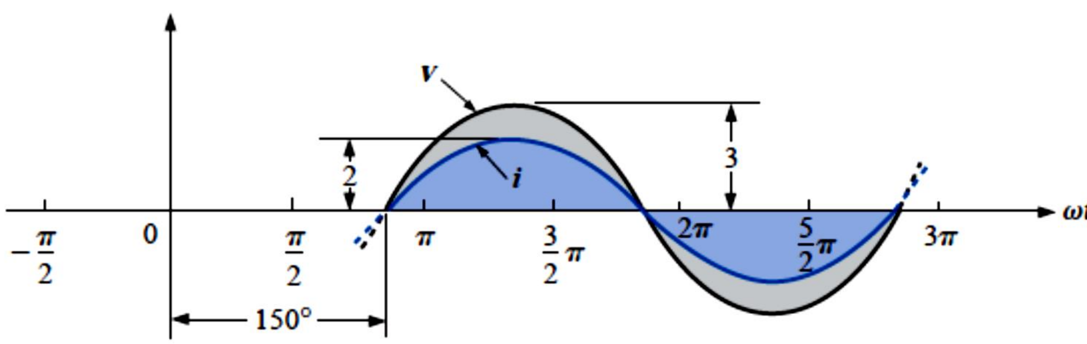


Figure 13: Example 4 (e); *v* and *i* are in phase

Average Value:

The average value of *any* current or voltage is the value indicated on a dc meter. In other words, over a complete cycle, the average value is the equivalent dc value. In general the average value of a waveform is given as:

$$G\ (Average\ value) = \frac{Area\ under\ the\ curve}{lemgth\ of\ the\ curve}$$

Example 5: Determine the average value of the waveforms of Fig. 14.

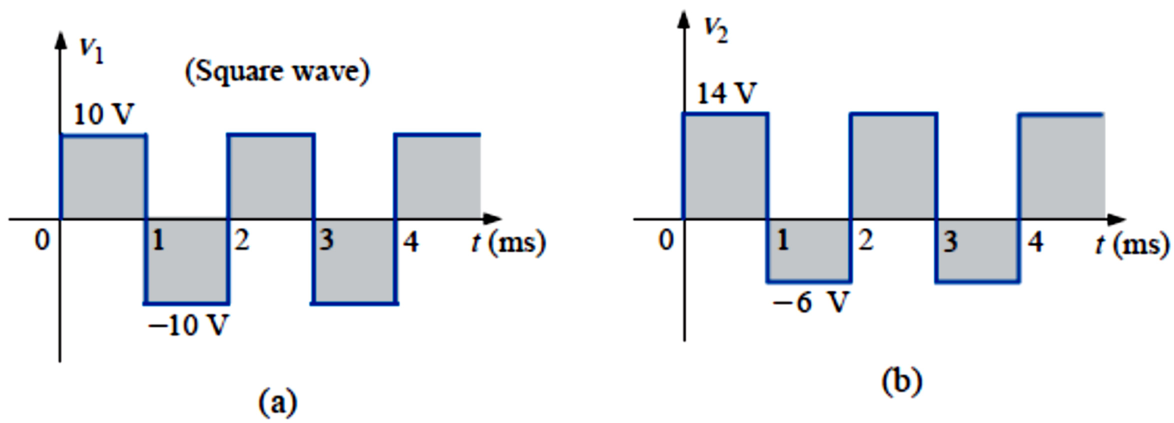


Figure 14: Example 5

Solution:

a. The area above the axis equals the area below over one cycle, resulting in an average value of zero volts. Using the above average equation:

$$G = \frac{(10\ \text{V})(1\ \text{ms}) - (10\ \text{V})(1\ \text{ms})}{2\ \text{ms}}$$

$$= \frac{0}{2\ \text{ms}} = \mathbf{0\ V}$$

b. By using the average equation:

$$G = \frac{(14\ \text{V})(1\ \text{ms}) - (6\ \text{V})(1\ \text{ms})}{2\ \text{ms}}$$

$$= \frac{14\ \text{V} - 6\ \text{V}}{2} = \frac{8\ \text{V}}{2} = \mathbf{4\ V}$$

In reality, the waveform of Fig. 14(b) is simply the square wave of Fig. 14(a) with a dc shift of 4 V; that is,

$v_2 = v_1 + 4\text{ V}$

as shown in Fig. 15.

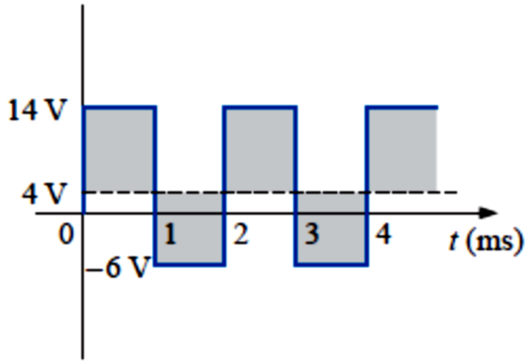


Figure 15: Defining the average value for the waveform of Fig. 14(b).

<u>Example 6:</u> Find the average values of the following waveforms over one full cycle:

a. Fig. 16.

b. Fig. 17.

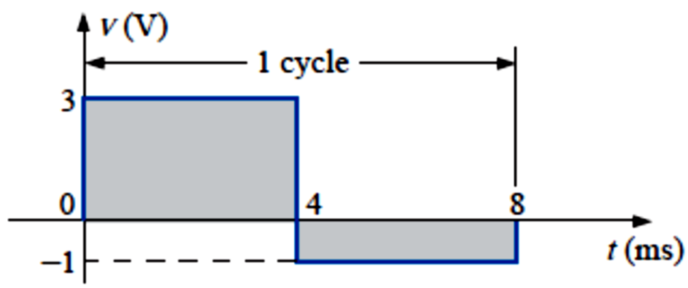


Figure 16: Example 6, Part a.

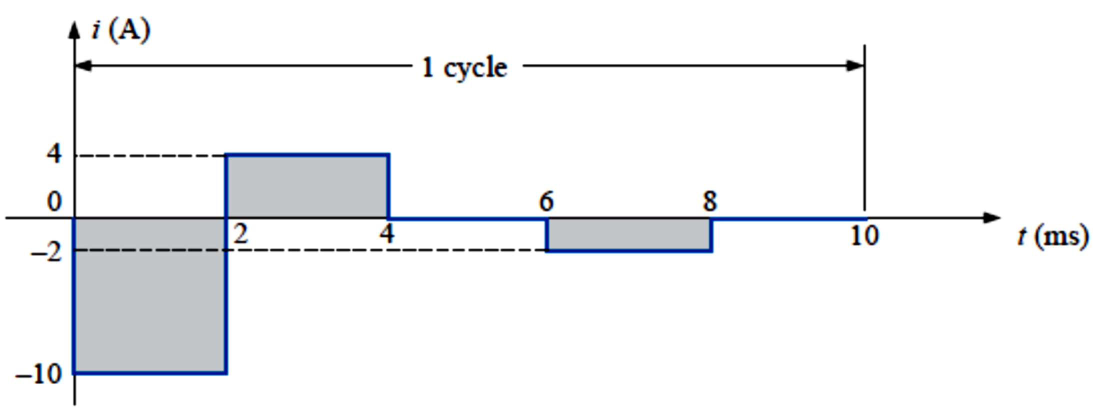


Figure 17: Example 6, Part b.

Solution:

$$\text{a. } G = \frac{+(3\text{ V})(4\text{ ms}) - (1\text{ V})(4\text{ ms})}{8\text{ ms}} = \frac{12\text{ V} - 4\text{ V}}{8} = \mathbf{1\text{ V}}$$

b. $$G = \frac{-(10\text{ V})(2\text{ ms}) + (4\text{ V})(2\text{ ms}) - (2\text{ V})(2\text{ ms})}{10\text{ ms}}$$

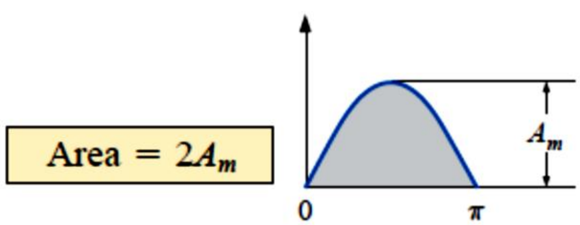
$$= \frac{-20\text{ V} + 8\text{ V} - 4\text{ V}}{10} = -\frac{16\text{ V}}{10} = \mathbf{-1.6\ V}$$

In the preceding examples, we found the areas under the curves by using a simple geometric formula. If we should encounter a sine wave or any other unusual shape, however, we must find the area by some other means (Integration over the specified period of time). Finding the area under the positive pulse of a sine wave using integration, we have:

$$\text{Area} = \int_0^{\pi} A_m \sin \alpha \, d\alpha$$

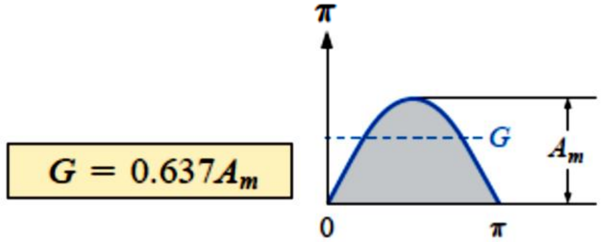
where ∫ is the sign of integration, **0** and **π** are the limits of integration,($\boldsymbol{A_m}$ **sin α**)is the function to be integrated, and (***dα***) indicates that we are integrating with respect to **α**. Integrating, we obtain:

$$\begin{aligned} \text{Area} &= A_m[-\cos \alpha]_0^{\pi} \\ &= -A_m(\cos \pi - \cos 0°) \\ &= -A_m[-1 - (+1)] = -A_m(-2) \end{aligned}$$



Since we know the area under the positive (or negative) pulse, we can easily determine the average value of the positive (or negative) region of a sine wave pulse by applying the average equation:

$$G = \frac{2A_m}{\pi}$$



<u>**Example 7:**</u> Determine the average value of the sinusoidal waveform of Fig.18.

Solution:

It is fairly obvious that:

the average value of a pure sinusoidal waveform over one full cycle is zero.

$$G = \frac{+2A_m - 2A_m}{2\pi} = \mathbf{0\ V}$$

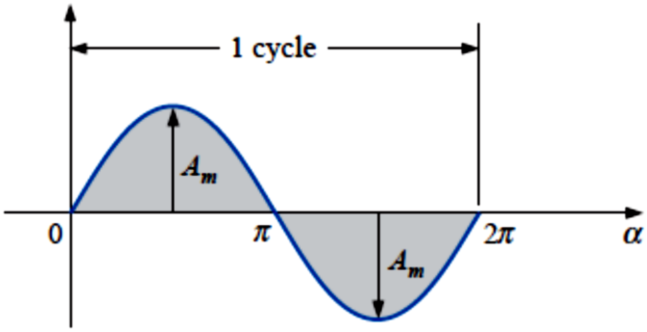


Figure 18: Example 7

<u>Example 8:</u> Determine the average value of the waveform of Fig. 19.

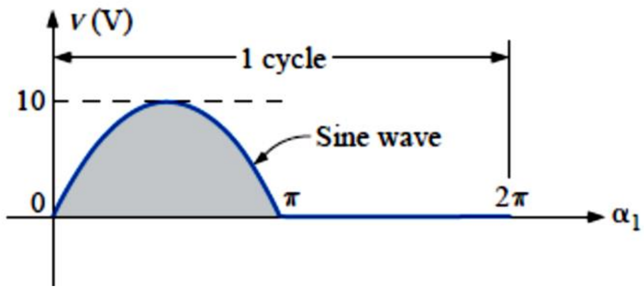


Figure 19: Example 8

Solution:

Determine the average value of the waveform of 8.

$$G = \frac{2A_m + 0}{2\pi} = \frac{2(10\ \text{V})}{2\pi} \cong \mathbf{3.18\ V}$$

<u>Effective (Root Mean Square RMS) Values</u>

An ***rms*** value is also known as the *effective* value and is defined in terms of the equivalent heating effect of direct current. The ***rms*** value of a sinusoidal voltage (or any time-varying voltage) is equivalent to the value of a dc voltage that causes an equal amount of heat (power dissipation) due to the circuit current flowing through a resistance.

The power delivered by the ac supply at any instant of time is:

$$P_{ac} = (i_{ac})^2R = (I_m \sin \omega t)^2R = (I_m^2 \sin^2\omega t)R$$

but

$$\sin^2 \omega t = \frac{1}{2}(1 - \cos 2\omega t) \qquad \text{(trigonometric identity)}$$

Therefore

$$P_{ac} = I_m^2\left[\frac{1}{2}(1 - \cos 2\omega t)\right]R$$

$$\boxed{P_{ac} = \frac{I_m^2 R}{2} - \frac{I_m^2 R}{2} \cos 2\omega t}$$

The equivalent dc value of a sinusoidal current or voltage is $\frac{1}{\sqrt{2}}$ or 0.707 of its maximum value. The equivalent dc value is called the effective value of the sinusoidal quantity.

In summary

$$\boxed{I_{eq(dc)} = I_{eff} = 0.707 I_m}$$

$$\boxed{I_m = \sqrt{2} I_{eff} = 1.414 I_{eff}}$$

$$\boxed{E_{eff} = 0.707 E_m}$$

$$\boxed{E_m = \sqrt{2} E_{eff} = 1.414 E_{eff}}$$

The effective value of any quantity plotted as a function of time can be found by using the following equation derived from the experiment just described:

$$I_{eff} = \sqrt{\frac{\int_0^T i^2(t)dt}{T}}$$

or

$$I_{eff} = \sqrt{\frac{area(i^2(t))}{T}}$$

Example 9: Find the *rms* values of the sinusoidal waveform in each part of Fig.20.

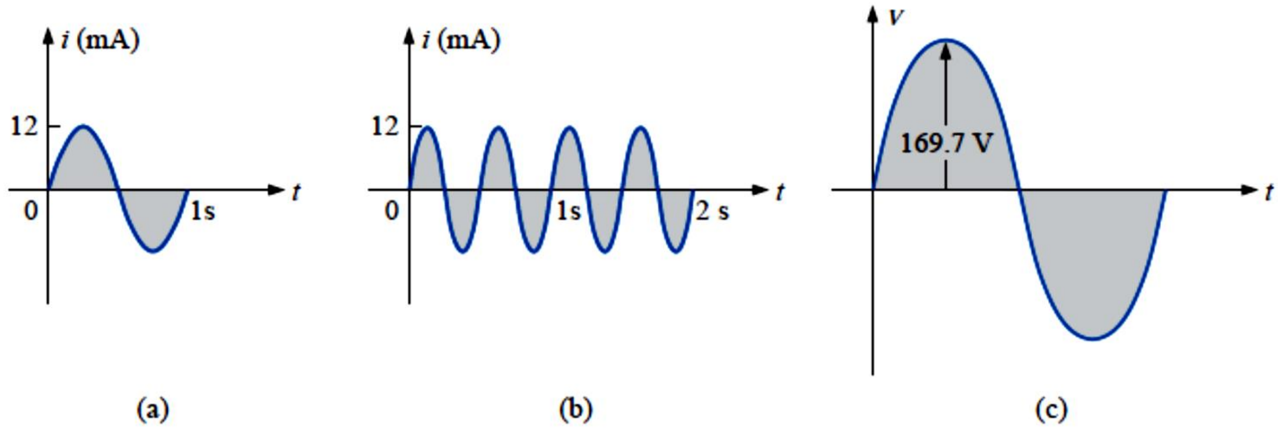


Figure 20: Example 9

Solution:

For part (a), $I_{rms} = 0.707(12 \times 10^{-3}\text{ A}) =$ **8.484 mA.**

For part (b), again $I_{rms} = 0.707(12 \times 10^{-3}\text{ A}) =$ **8.484 mA.** Note that frequency did not change the effective value in (b) above compared to (a).

For part (c), $V_{rms} = 0.707(169.73\text{ V}) =$ **120 V.**

Example 10: The 120-V dc source of Fig. 21(a) delivers 3.6 W to the load. Determine the peak value of the applied voltage (*Em*) and the current (*Im*) if the ac source [Fig. 21(b)] is to deliver the same power to the load.

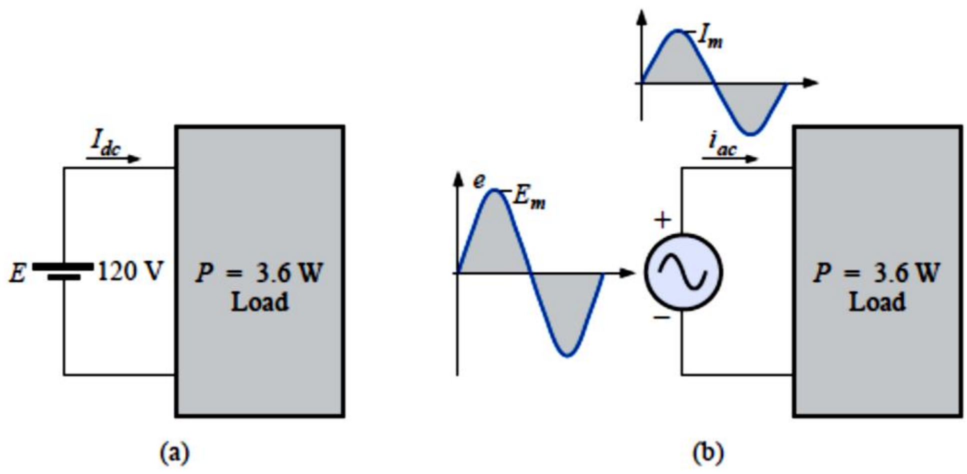


Figure 21: Example 10

Solution:

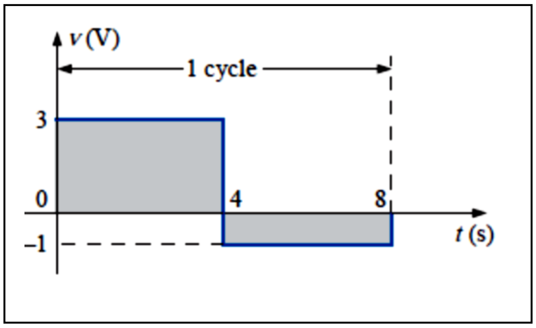
$$P_{dc} = V_{dc} I_{dc}$$

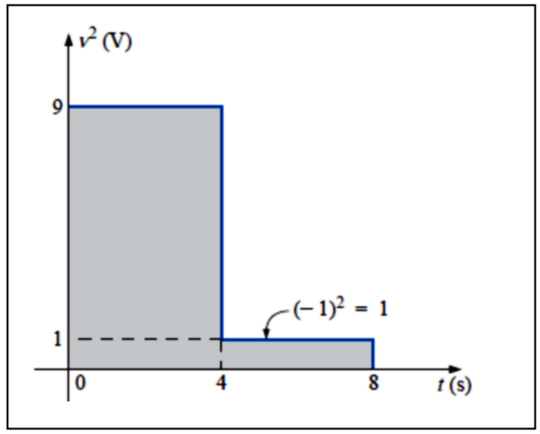
$$I_{dc} = \frac{P_{dc}}{V_{dc}} = \frac{3.6W}{120V} = 30mA$$

$$I_m = \sqrt{2}\, I_{dc} = (1.414)(30mA) = 42.42\, mA$$

$E_m = \sqrt{2}\,E_{dc} = (1.414)(120V) = 169.68\,V$

<u>Example 11:</u> Find the effective or *rms* value of the waveform of Fig. 22.





(a) **(b)**

Figure 22: Example 11, (a) the waveform, (b) The squared waveform in (a)

Solution:

$$V_{rms} = \sqrt{\frac{(9)(4)+(1)(4)}{8}} = \sqrt{\frac{40}{8}} = \mathbf{2.236\ V}$$

<u>Example 12:</u> Calculate the *rms* value of the voltage of Fig. 23.

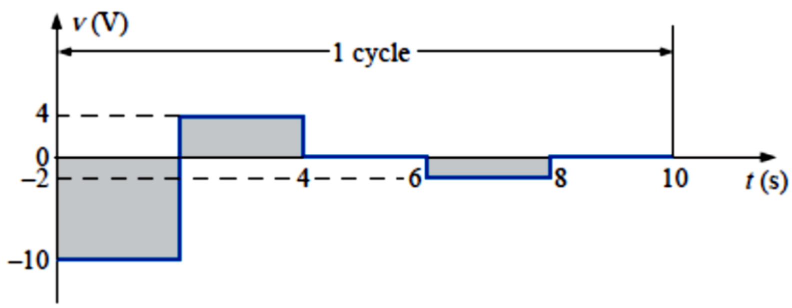


Figure 23: Example 12

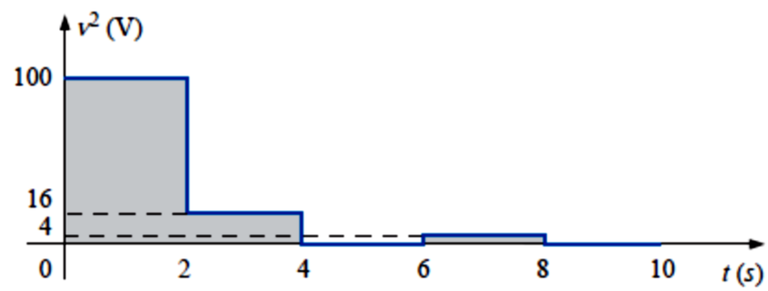


Figure 24: The squared waveform of Fig. 23.

Solution:

$$V_{\mathrm{rms}} = \sqrt{\frac{(100)(2) + (16)(2) + (4)(2)}{10}} = \sqrt{\frac{240}{10}}$$

$$= \mathbf{4.899\ V}$$

Example 13: Determine the average and rms values of the square wave of Fig.25.

Solution: By inspection, the average value is zero.

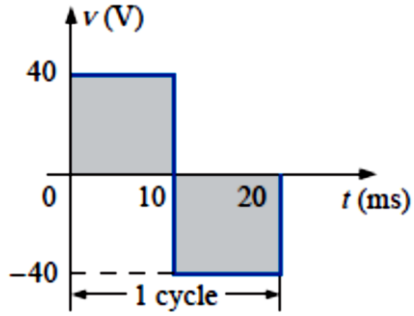


Figure 25: Example 13

$$V_{\mathrm{rms}} = \sqrt{\frac{(1600)(10 \times 10^{-3}) + (1600)(10 \times 10^{-3})}{20 \times 10^{-3}}}$$

$$= \sqrt{\frac{32{,}000 \times 10^{-3}}{20 \times 10^{-3}}} = \sqrt{1600}$$

$$V_{\mathrm{rms}} = \mathbf{40\ V}$$

Example 14: Find the amplitude, phase, period, and frequency of the sinusoid:

$$v(t) = 12\cos(50t + 10^\circ)$$

Solution:

The amplitude is $V_m = 12$ V.
The phase is $\phi = 10^\circ$.
The angular frequency is $\omega = 50$ rad/s.

$\omega = 2\pi f = 50$

$\therefore\ f = \frac{50}{2\pi} = 7.958$ Hz

Example 15 (H.W.): Given the sinusoid 5 sin ($4\pi t$ - 60^{o}), calculate its amplitude, phase, angular frequency, period, and frequency.

Answer: 5, $-60°$, 12.57 rad/s, 0.5 s, 2 Hz.

Phasors:

Sinusoids are easily expressed in terms of phasors, which are more convenient to work with than sine and cosine functions.

A phasor is a complex number that represents the amplitude and phase of a sinusoid.

Phasors provide a simple means of analyzing linear circuits excited by sinusoidal sources; A complex number z can be written in rectangular form as:

$$z = x + jy$$

where $j = \sqrt{-1}$; x is the real part of z; y is the imaginary part of z. In this context, the variables x and y do not represent a location as in two-dimensional vector analysis but rather the real and imaginary parts of z in the complex plane.

The complex number z can also be written in polar or exponential form as:

$$z = r\underline{/\phi} = re^{j\phi}$$

where r is the magnitude of z, and ϕ is the phase of z. We notice that z can be represented in three ways:

$z = x + jy$	Rectangular form
$z = r\underline{/\phi}$	Polar form
$z = re^{j\phi}$	Exponential form

The relationship between the rectangular form and the polar form is shown in Fig. 26,

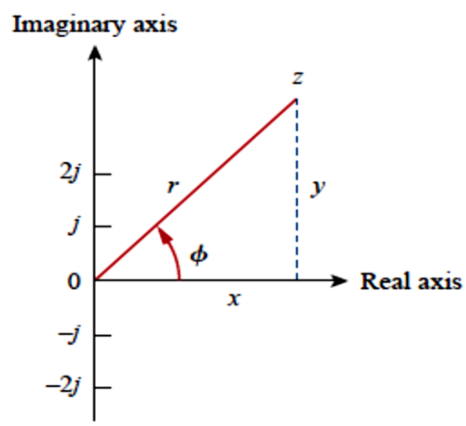


Figure 26: Representation of a complex number $z = x + jy = r\underline{/\phi}$.

where the x axis represents the real part and the y axis represents the imaginary part of a complex number. Given x and y, we can get r and ϕ as:

$$r = \sqrt{x^2 + y^2}, \qquad \phi = \tan^{-1}\frac{y}{x}$$

On the other hand, if we know r and ϕ we can obtain x and y as:

$$x = r\cos\phi, \qquad y = r\sin\phi$$

Thus, z may be written as:

$$z = x + jy = r\underline{/\phi} = r(\cos\phi + j\sin\phi)$$

Addition and subtraction of complex numbers are better performed in rectangular form; multiplication and division are better done in polar form. Given the complex numbers:

$$z = x + jy = r\underline{/\phi}, \qquad z_1 = x_1 + jy_1 = r_1\underline{/\phi_1}$$

$$z_2 = x_2 + jy_2 = r_2\underline{/\phi_2}$$

the following operations are important.

- **Addition:**

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$$

- **Subtraction:**

$$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$$

- **Multiplication:**

$$z_1 z_2 = r_1 r_2\underline{/\phi_1 + \phi_2}$$

- **Division:**

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\underline{/\phi_1 - \phi_2}$$

- **Reciprocal:**

$$\frac{1}{z} = \frac{1}{r}\underline{/-\phi}$$

- **Square Root:**

$$\sqrt{z} = \sqrt{r}\,\underline{/\phi/2}$$

- **Complex Conjugate:**

$$z^* = x - jy = r\underline{/-\phi} = re^{-j\phi}$$

The idea of phasor representation is based on Euler's identity. In general,

$$e^{\pm j\phi} = \cos\phi \pm j\sin\phi$$

which shows that we may regard cos ϕ and sin ϕ as the real and imaginary parts of $e^{j\emptyset}$ we may write:

$$\cos\phi = \mathrm{Re}(e^{j\phi})$$
$$\sin\phi = \mathrm{Im}(e^{j\phi})$$

where Re and Im stand for the real part of and the imaginary part of.

Example 16: Evaluate these complex numbers:

(a) $(40\angle 50^\circ + 20\angle -30^\circ)^{1/2}$

(b) $\dfrac{10\angle -30^\circ + (3 - j4)}{(2 + j4)(3 - j5)^*}$

Solution:

(a) Using polar to rectangular transformation,

$$40\angle 50^\circ = 40(\cos 50^\circ + j\sin 50^\circ) = 25.71 + j30.64$$
$$20\angle -30^\circ = 20[\cos(-30^\circ) + j\sin(-30^\circ)] = 17.32 - j10$$

Adding them up gives

$$40\angle 50^\circ + 20\angle -30^\circ = 43.03 + j20.64 = 47.72\angle 25.63^\circ$$

Taking the square root of this,

$$(40\angle 50^\circ + 20\angle -30^\circ)^{1/2} = 6.91\angle 12.81^\circ$$

(b) Using polar-rectangular transformation, addition, multiplication, and division,

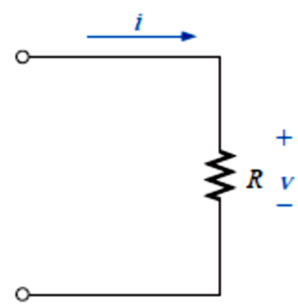
$$\frac{10\angle -30^\circ + (3 - j4)}{(2 + j4)(3 - j5)^*} = \frac{8.66 - j5 + (3 - j4)}{(2 + j4)(3 + j5)}$$
$$= \frac{11.66 - j9}{-14 + j22} = \frac{14.73\angle -37.66^\circ}{26.08\angle 122.47^\circ}$$
$$= 0.565\angle -160.13^\circ$$

Response Of Basic R, L, And C Elements to A Sinusoidal Voltage or Current

- **Resistor**

For $v = V_m \sin \omega t$,

$$i = \frac{v}{R} = \frac{V_m \sin \omega t}{R} = \frac{V_m}{R} \sin \omega t = I_m \sin \omega t$$



Where:

$$I_m = \frac{V_m}{R}$$

In addition, for a given i,

$$v = iR = (I_m \sin \omega t)R = I_m R \sin \omega t = V_m \sin \omega t$$

Where:

$$V_m = I_m R$$

A plot of v and i in Fig. 27 reveals that:

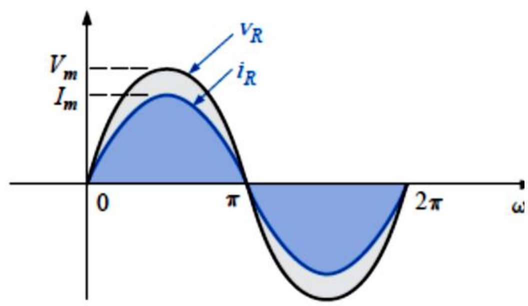


Figure 27: The voltage and current of a resistive element are in phase.

for a purely resistive element, the voltage across and the current through the element are in phase, with their peak values related by Ohm's law.

- **Inductor**

For the inductor of Fig. 28:

$$v_L = L\frac{di_L}{dt}$$

and, applying differentiation,

$$\frac{di_L}{dt} = \frac{d}{dt}(I_m \sin \omega t) = \omega I_m \cos \omega t$$

Therefore:

$$v_L = L\frac{di_L}{dt} = L(\omega I_m \cos \omega t) = \omega L I_m \cos \omega t$$

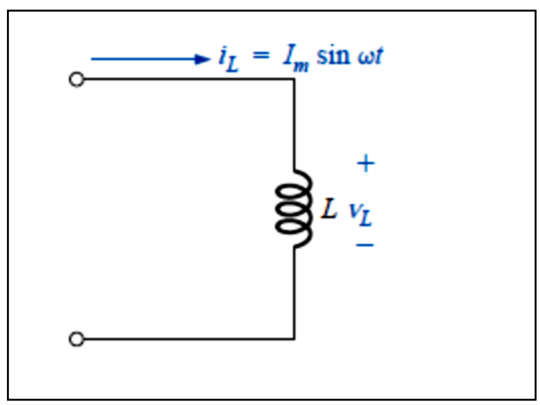


Figure 28: Investigating the sinusoidal response of an inductive element.

Or

$v_L = V_m \sin(\omega t + 90°)$

Where:

$V_m = \omega L I_m$

Note that the peak value of v_L is directly related to ω (= 2πf) and L

A plot of v_L and i_L in Fig. 29 reveals that:

for an inductor, v_L leads i_L by 90°, or i_L lags v_L by 90°.

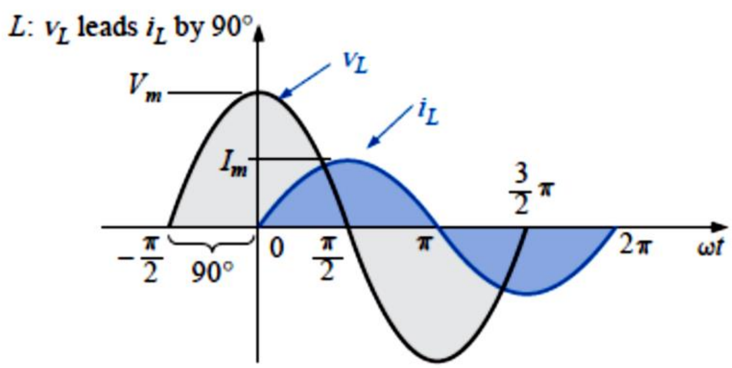


Figure 29: For a pure inductor, the voltage across the coil leads the current through the coil by 90°.

The quantity ωL, called the reactance (from the word reaction) of an inductor, is symbolically represented by X_L and is measured in ohms, and L is measured in Henry (H); that is,

$$\boxed{X_L = \omega L} \qquad \text{(ohms, } \Omega\text{)}$$

In an Ohm's law format, its magnitude can be determined from:

$$\boxed{X_L = \frac{V_m}{I_m}} \qquad \text{(ohms, } \Omega\text{)}$$

- **Capacitor**

For the capacitor of Fig. 30,

$$i_C = C\,\frac{dv_C}{dt}$$

and, applying differentiation,

$$\frac{dv_C}{dt} = \frac{d}{dt}(V_m \sin \omega t) = \omega V_m \cos \omega t$$

Therefore:

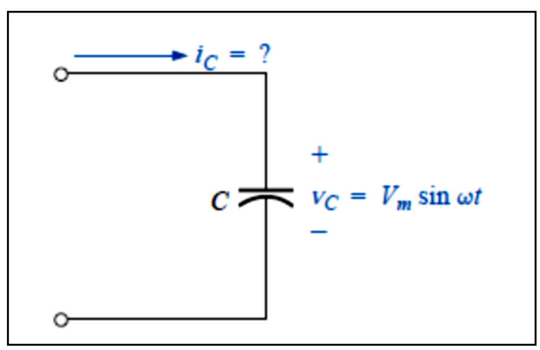


Figure 30: Investigating the sinusoidal response of an capacitive element.

$$i_C = C\frac{dv_C}{dt} = C(\omega V_m \cos \omega t) = \omega C V_m \cos \omega t$$

Or:

$$i_C = I_m \sin(\omega t + 90°)$$

Where:

$$I_m = \omega C V_m$$

Note that the peak value of i_C is directly related to ω (= 2πf) and C, as predicted in the discussion above.

A plot of v_C and i_C in Fig. 31 reveals that:

for a capacitor, i_C leads v_C by 90°, or v_C lags i_C by 90°.

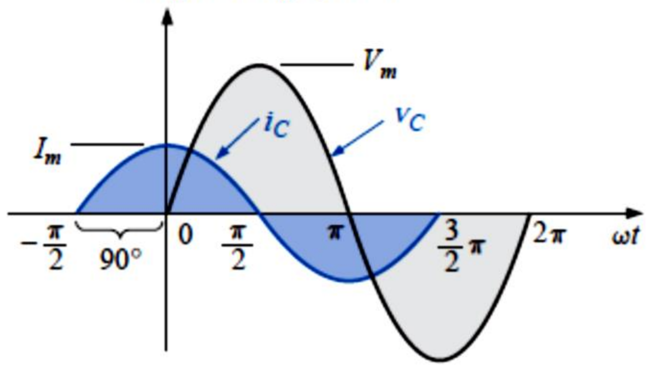


Figure 31: The current of a purely capacitive element leads the voltage across the element by 90°.

The quantity 1/ωC, called the reactance of a capacitor, is symbolically represented by X_C and is measured in ohms, and C in Farad (F); that is,

$$\boxed{X_C = \frac{1}{\omega C}} \quad \text{(ohms, } \Omega\text{)}$$

In an Ohm's law format, its magnitude can be determined from:

$$\boxed{X_C = \frac{V_m}{I_m}} \quad \text{(ohms, } \Omega\text{)}$$

<u>Example 17:</u> The voltage across a resistor is indicated. Find the sinusoidal expression for the current if the resistor is 10Ω. Sketch the curves for *v* and *i*.

a. $v = 100 \sin 377t$

b. $v = 25 \sin(377t + 60°)$

Solution:

a.

$$I_m = \frac{V_m}{R} = \frac{100\text{ V}}{10\ \Omega} = 10\text{ A}$$

(v and i are in phase), resulting in

$$i = \mathbf{10 \sin 377}t$$

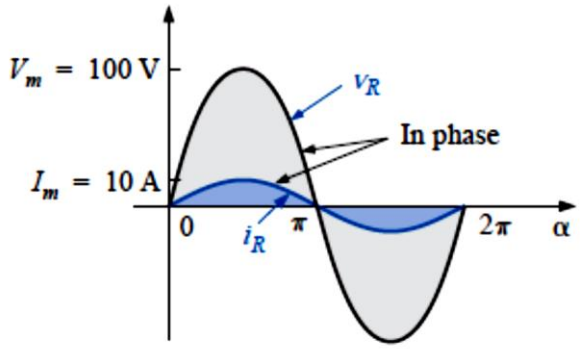


Figure 32: Example 17(a).

b.

$$I_m = \frac{V_m}{R} = \frac{25\text{ V}}{10\ \Omega} = 2.5\text{ A}$$

(v and i are in phase), resulting in

$$i = \mathbf{2.5 \sin(377}t \mathbf{+ 60°)}$$

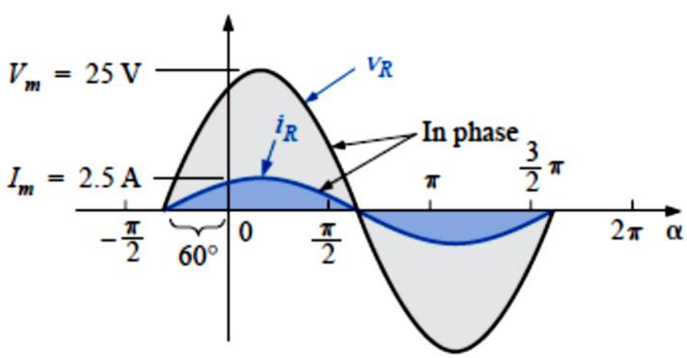


Figure 33: Example 17(b).

<u>Example 18:</u> The current through a 5Ω resistor is given. Find the sinusoidal expression for the voltage across the resistor for $i = 40 \sin(377t + 30°)$.

Solution: $V_m = I_m R = (40\text{ A})(5\Omega) = 200\text{ V}$

(v and i are in phase), resulting in

$$v = \mathbf{200 \sin(377}t + \mathbf{30°)}$$

<u>Example 19:</u> The current through a 0.1-H coil is provided. Find the sinusoidal expression for the voltage across the coil. Sketch the v and i curves.

a. $i = 10 \sin 377t$

b. $i = 7 \sin(377t - 70°)$

Solution:

a.

$X_L = \omega L = (377 \text{ rad/s})(0.1 \text{ H}) = 37.7\ \Omega$

$V_m = I_m X_L = (10 \text{ A})(37.7\ \Omega) = 377 \text{ V}$

and we know that for a coil *v* leads *i* by 90°. Therefore,

$\mathbf{v = 377 \sin(377t + 90°)}$

The curves are sketched in Fig. 34.

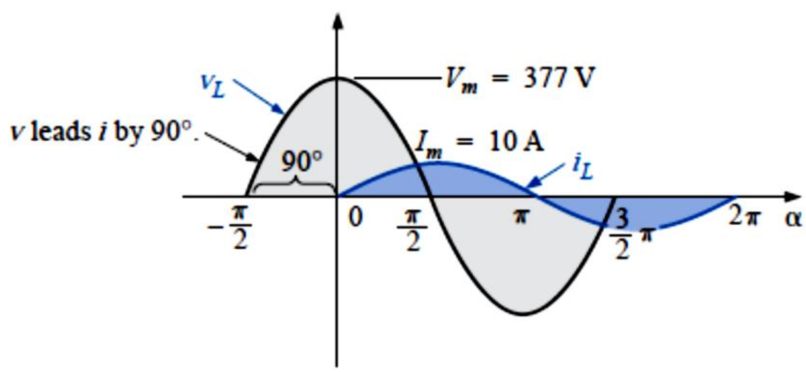


Figure 34: Example 19(a).

b.

X_L remains at 37.7Ω

$V_m = I_m X_L = (7 \text{ A})(37.7\ \Omega) = 263.9 \text{ V}$.

and we know that for a coil *v* leads *i* by 90°. Therefore,

$\mathbf{v = 263.9 \sin(377t + 20°)}$

The curves are sketched in Fig. 35

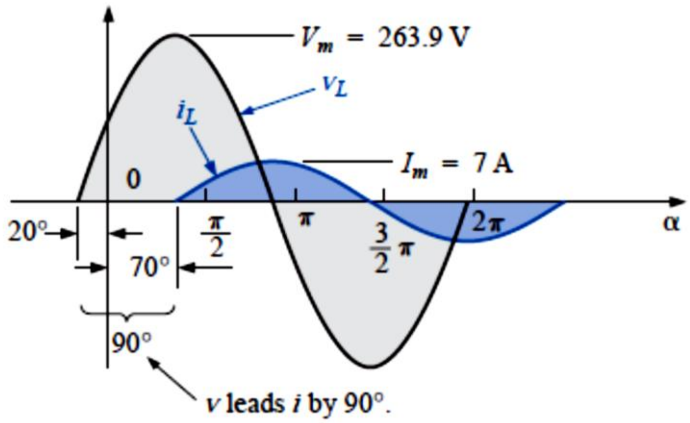


Figure 35: Example 19(b).

Example 20: The voltage across a 0.5-H coil is provided below. What is the sinusoidal expression for the current?

$$v = 100 \sin 20t$$

Solution:

$$X_L = \omega L = (20 \text{ rad/s})(0.5 \text{ H}) = 10 \ \Omega$$

$$I_m = \frac{V_m}{X_L} = \frac{100 \text{ V}}{10 \ \Omega} = 10 \text{ A}$$

and we know that i lags v by 90°. Therefore,

$$\boldsymbol{i = 10 \sin(20t - 90°)}$$

Example 21: The voltage across a 1μF capacitor is provided below. What is the sinusoidal expression for the current? Sketch the *v* and *i* curves.

$$v = 30 \sin 400t$$

Solution:

$$X_C = \frac{1}{\omega C} = \frac{1}{(400 \text{ rad/s})(1 \times 10^{-6} \text{ F})} = \frac{10^6 \ \Omega}{400} = 2500 \ \Omega$$

$$I_m = \frac{V_m}{X_C} = \frac{30 \text{ V}}{2500 \ \Omega} = 0.0120 \text{ A} = 12 \text{ mA}$$

and we know that for a capacitor *i* leads *v* by 90°. Therefore,

$$\boldsymbol{i = 12 \times 10^{-3} \sin(400t + 90°)}$$

The curves are sketched in Fig. 36

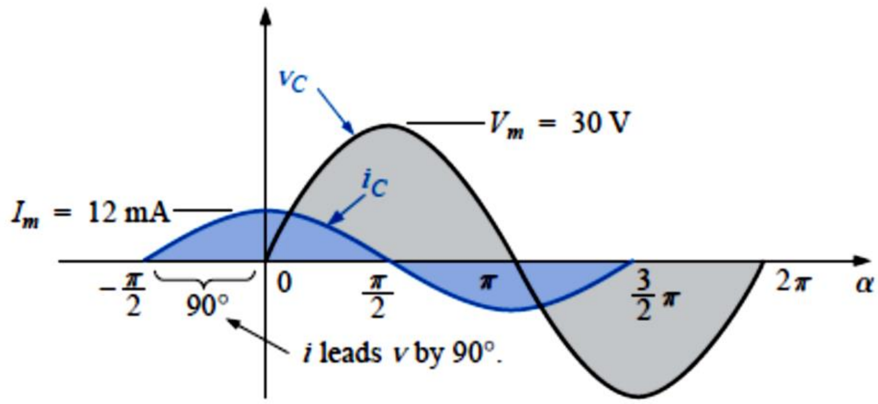


Figure 36: Example 21.

<u>Example 22:</u>The current through a 100μF capacitor is given. Find the sinusoidal expression for the voltage across the capacitor.

$$i = 40 \sin(500t + 60°)$$

Solution:

$$X_C = \frac{1}{\omega C} = \frac{1}{(500 \text{ rad/s})(100 \times 10^{-6} \text{ F})} = \frac{10^6\ \Omega}{5 \times 10^4} = \frac{10^2\ \Omega}{5} = 20\ \Omega$$

$$V_m = I_m X_C = (40 \text{ A})(20\ \Omega) = 800 \text{ V}$$

and we know that for a capacitor, v lags i by 90°. Therefore,

$$v = 800 \sin(500t + 60° - 90°)$$

and $$\mathbf{v = 800 \sin(500t - 30°)}$$

<u>Example 23:</u> For the following pairs of voltages and currents, determine whether the element involved is a capacitor, an inductor, or a resistor, and determine the value of C, L, or R if sufficient data are provided (Fig. 37):

a. $v = 100 \sin(\omega t + 40°)$
   $i = 20 \sin(\omega t + 40°)$

b. $v = 1000 \sin(377t + 10°)$
   $i = 5 \sin(377t - 80°)$

c. $v = 500 \sin(157t + 30°)$
   $i = 1 \sin(157t + 120°)$

d. $v = 50 \cos(\omega t + 20°)$
   $i = 5 \sin(\omega t + 110°)$

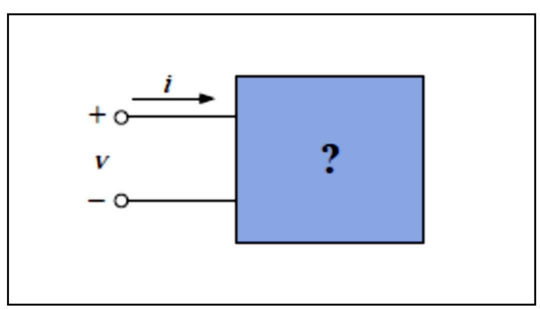


Figure 37: Example 23.

Solution:

a. Since v and i are *in phase,* the element is a *resistor,* and

$$R = \frac{V_m}{I_m} = \frac{100\text{ V}}{20\text{ A}} = \mathbf{5\ \Omega}$$

b. Since v *leads* i by 90°, the element is an *inductor,* and

$$X_L = \frac{V_m}{I_m} = \frac{1000\text{ V}}{5\text{ A}} = 200\ \Omega$$

so that $X_L = \omega L = 200\ \Omega$ or

$$L = \frac{200\ \Omega}{\omega} = \frac{200\ \Omega}{377\text{ rad/s}} = \mathbf{0.531\ H}$$

c. Since i *leads* v by 90°, the element is a *capacitor,* and

$$X_C = \frac{V_m}{I_m} = \frac{500\text{ V}}{1\text{ A}} = 500\ \Omega$$

so that $X_C = \dfrac{1}{\omega C} = 500\ \Omega$ or

$$C = \frac{1}{\omega 500\ \Omega} = \frac{1}{(157\text{ rad/s})(500\ \Omega)} = \mathbf{12.74\ \mu F}$$

d. $v = 50\cos(\omega t + 20°) = 50\sin(\omega t + 20° + 90°)$
$= 50\sin(\omega t + 110°)$

Since v and i are *in phase,* the element is a *resistor,* and

$$R = \frac{V_m}{I_m} = \frac{50\text{ V}}{5\text{ A}} = \mathbf{10\ \Omega}$$